A Solution of the Cauchy Problem for Multidimensional Discrete Linear Shift-Invariant Systems*

Sandro Zampieri
Department of Electronics and Informatics
University of Padova
via Gradenigo 6/A
35131 Padova, Italy

Submitted by Richard A. Brualdi

ABSTRACT

We propose a solution of the Cauchy problem for multidimensional linear shift-invariant systems over $\mathbf{Z}^r$ that can be described by a set of multidimensional difference equations. We give a method for finding an initial condition structure. It consists in finding subsets of $\mathbf{Z}^r$ where the signals can be given arbitrarily and where they fix the trajectory on the whole grid. Moreover we develop an efficient algorithm computing the whole trajectory, once given the initial conditions. For this purpose we develop some constructive techniques for Laurent polynomial modules.

1. INTRODUCTION

It is well known that Gröbner basis theory for polynomial ideals and modules is very important in computer algebra, since it has produced many efficient algorithms for symbolic manipulation of algebraic formulas. Recently other applications have been considered in multidimensional system theory, both in the input-output and in the behavioral approach [1–3]. In particular, several applications of Gröbner basis techniques in this context have been

* After the present paper had been sent to the editors, I learned of Oberst and Zerz's article, "The canonical Cauchy problem for linear systems of partial difference equations with constant coefficients over the complete r-dimensional integral lattice $\mathbf{Z}^r$," which has been submitted for publication. In this paper the authors use a more abstract formalism, but the solution of the Cauchy problem they propose seems substantially similar to ours.

655 Avenue of the Americas, New York, NY 10010 0024-3795/94/$7.00

discussed in [4]. Another example of the possible applications is given in [5], where Gröbner basis algorithms are needed for analyzing feedback stabilizability of a 2D system.

Gröbner basis techniques have been extensively used also in the behavioral approach to multidimensional systems. In this setting [6] a dynamical system Σ is defined as a triple

$$\Sigma = (T, W, \mathscr{B}),$$

where T is the time set, W is the signal alphabet, and $\mathscr{B} \subseteq W^T$ is the behavior. So the behavior describes the dynamics by specifying the time signals that the system can conceivably generate.

A multidimensional discrete linear shift-invariant system is characterized by the time set $T = \mathbf{N}^r$ or $T = \mathbf{Z}^r$, by the signal alphabet $W = F^q$, where F is a field and q a positive integer, and by a behavior $\mathscr{B}$ which coincides with the set of solutions of a family of difference equations. Therefore the study of multidimensional difference equations is a necessary step in developing a theory of multidimensional systems in the behavioral approach. One important problem in this field is the computation of solutions of a family of multidimensional difference equations starting from some initial conditions. More precisely, the set solutions $\mathscr{B} \subseteq (F^q)^T$ of a family of difference equations has in general infinitely many elements. However, it can be showed that there exist subsets $G_1, \dots, G_q$ of T with the following property: for every $z_i \in F^{G_i}$, $i = 1, \dots, q$, there exists a unique solution $w = (w_1, \dots, w_q) \in \mathscr{B}$ such that $w_{i|G_i} = z_i$. In other words, each component w_i of the solution $w \in \mathscr{B}$ can be fixed arbitrarily in G_i, and moreover there are no other solutions satisfying the same initial conditions. The problem of finding subsets G_i with the previous property and of constructing the solution starting from the initial conditions is called the Cauchy problem. Note that if $G_i = T$ for some $i \in \{1, \dots, q\}$, we have that the ith component of the solution is free, since it can be fixed arbitrarily.

In [1, 2] the Gröbner basis techniques are used to obtain a canonical form of a multidimensional system when $T = \mathbf{Z}^2$. Such a representation gives the instruments for analyzing the properties of the system and, above all, for determining the free variables and the initial condition structure. Oberst in [3] solves the same problems for multidimensional systems over $\mathbf{N}^r$. A direct extension of his techniques to multidimensional systems defined over $\mathbf{Z}^r$ is possible only for the extraction of the free variable structure. On the contrary, the Cauchy problem of finding the initial condition structure and of computing the trajectories has no immediate extension.

In this paper we present such an extension, based on the introduction of algorithmic techniques for free modules over the Laurent polynomial ring $F[x_1, \ldots, x_r, x_1^{-1}, \ldots, x_r^{-1}]$, i.e. the ring of all the polynomials in which both positive and negative powers of the indeterminates are allowed. The central result is an isomorphism between the Laurent polynomial ring $F[x_1, \ldots, x_r, x_1^{-1}, \ldots, x_r^{-1}]$ and the quotient ring $F[x_1, \ldots, x_r, y_1, \ldots, y_r]/\mathscr{U}$, where $\mathscr{U}$ is the ideal generated by the polynomials $x_1 y_1 - 1, \ldots, x_r y_r - 1$. By resorting to this isomorphism, we can extend the Gröbner basis techniques to Laurent polynomial modules, and in particular we can solve the Cauchy problem for multidimensional systems over $\mathbf{Z}^r$.

2. CONSTRUCTIVE TECHNIQUES FOR LAURENT POLYNOMIAL MODULES

Let F be a field, and $F[\mathbf{x}] := F[x_1, \ldots, x_r]$ the ring of polynomials in the r indeterminates $x_1, \ldots, x_r$, with coefficients in F. Let $T(\mathbf{x})$ be the monoid of all the monic monomials in $x_1, \ldots, x_r$.

Consider the $F[\mathbf{x}]$-module $F[\mathbf{x}]^p$, i.e. the module of the p-tuples $(f_1, \ldots, f_p)^T$, $f_i \in F[\mathbf{x}]$, and for any polynomial vector f in $F[\mathbf{x}]^p$ let $T(\mathbf{x})f := \{vf : v \in T(\mathbf{x})\}$. Denoting by $e_1, \ldots, e_p$ the canonical generators of $F[\mathbf{x}]^p$, we introduce the subset $T_p(\mathbf{x})$ of $F[\mathbf{x}]^p$ as follows:

$$T_p(\mathbf{x}) := \bigcup_{i=1}^{p} T(\mathbf{x}) e_i = \left\{ (0, \ldots, 0, \overset{i}{t}, 0, \ldots, 0)^T : i = 1, \ldots, p, t \in T(\mathbf{x}) \right\}.$$

We will introduce now some concepts connected with the Gröbner basis for submodules of $F[\mathbf{x}]^p$. We will follow the approach proposed in [7] by Möller and Mora.

Consider in $T_p(\mathbf{x})$ a term ordering $<_T$, i.e. a total ordering which satisfies the following two properties:

1. $t <_T vt$ for all $t \in T_p(\mathbf{x})$ and $v \in T(\mathbf{x})$;
2. $t_1 <_T t_2$ implies $vt_1 <_T vt_2$ for all $v \in T(\mathbf{x})$.

Since $T_p(\mathbf{x})$ is a basis of the F-vector space $F[\mathbf{x}]^p$, then for any $0 \neq f \in F[\mathbf{x}]^p$ there exists a unique representation

$$f = \sum_{i=1}^{s} c_i t_i,$$

where $0 \neq c_i \in F$, $t_i \in T_p(\mathbf{x})$, and $t_{i+1} <_T t_i$ for all i. The greatest term t_1 will be denoted as $\mathrm{Hterm}(f)$.

Given a submodule $\mathscr{M}$ of $F[\mathbf{x}]^p$, we say that a finite subset $\mathscr{G}$ of $\mathscr{M}$ is a Gröbner basis of $\mathscr{M}$ if, for all $0 \neq f \in \mathscr{M}$, a representation (G-representation)

$$f = \sum_{i=1}^{s} c_i v_i g_i,$$

exists, where $0 \neq c_i \in F$, $v_i \in T(\mathbf{x})$, $g_i \in \mathscr{G}$, and $v_{i+1}\,\mathrm{Hterm}(g_{i+1}) <_T v_i\,\mathrm{Hterm}(g_i)$ for all i.

It can be seen that a Gröbner basis always exists. Moreover there exist algorithms that provide a Gröbner basis $\mathscr{G} = \{g_1, \dots, g_n\}$ of $\mathscr{M}$ from any set $\{p_1, \dots, p_l\}$ of generators of $\mathscr{M}$, and provide also the polynomials $\delta_{i1}, \dots, \delta_{il}$ such that

$$g_i = \delta_{i1} p_1 + \cdots + \delta_{il} p_l,$$

for each $i = 1, \dots, n$.

EXAMPLE. Let $\mathscr{I}$ be the ideal in $F[x_1, x_2]$ generated by p_1, p_2, where $p_1 = x_1^2 x_2^2 - 3x_1^2$ and $p_2 = x_1 x_2^3 + 2x_1^2 - x_1 + 1$, and let $<_T$ be the term ordering in $T(x_1, x_2)$ defined as follows:

$$x_1^{h_1} x_2^{h_2} <_T x_1^{k_1} x_2^{k_2} \quad \text{if and only if} \quad h_1 < h_2 \text{ or } h_1 = k_1,\ h_2 < k_2.$$

Then, by the Gröbner basis construction algorithm, we obtain the Gröbner basis $\{g_1, g_2\}$ where $g_1 = x_1^2 + \frac{3}{2}x_1 x_2 - \frac{1}{2}x_1 + \frac{1}{2}$ and $g_2 = x_2^2 - 3$. We obtain moreover that

$$g_1 = \left(x_1 x_2 - \tfrac{1}{2}x_2^4 + \tfrac{1}{2}x_2\right)p_1 + \left(-\tfrac{1}{2}x_1 x_2^3 - \tfrac{3}{2}x_1 x_2 - \tfrac{1}{2}\right)p_2,$$

$$\begin{aligned} g_2 &= \left(2x_1 x_2^3 - 2x_1 + x_2^6 - 2x_2^3 - 1\right)p_1 \\ &\quad + \left(-x_1 x_2^5 + 3x_1 x_2^3 + x_1 x_2^2 - 3x_1 + x_2^2 - 3\right)p_2. \end{aligned}$$

It is clear that $\{p_1, p_2\}$ is not a Gröbner basis of $\mathscr{I}$ w.r.t. the term ordering $<_T$, since the polinomial $x_2^2 - 3$ admits no G-representations w.r.t. this set of generators.

Gröbner bases are important mainly because they allow one to solve in a constructive way many important problems in classical ideal theory. They play an important role also for the solution of the Cauchy problem when the difference equations operate on signals with support in $\mathbf{N}^r$. The following lemma and proposition, proved in [3], are the key results for the solution of the Cauchy problem and will be extended to Laurent polynomial submodules in order to solve the same problem in $\mathbf{Z}^r$.

LEMMA 1. *Let $\mathscr{M}$ be a submodule of $F[\mathbf{x}]^p$, and* $\mathrm{Hterm}(\mathscr{M}) := \{\mathrm{Hterm}(f) : 0 \neq f \in \mathscr{M}\}$. *Then*

$$F[\mathbf{x}]^p = \mathscr{M} \oplus F[\mathbf{x}]_T^p, \tag{1}$$

where $T = T_p(\mathbf{x}) \setminus \mathrm{Hterm}(\mathscr{M})$, $F[\mathbf{x}]_T^p$ is the F-subspace of $F[\mathbf{x}]^p$ generated by T, and $\oplus$ means direct sum of F-vector spaces.

PROPOSITION 1. *If $\mathscr{M}$ is a submodule of $F[\mathbf{x}]^p$, then $\mathscr{G} = \{g_1, \dots, g_n\}$ is a Gröbner basis of $\mathscr{M}$ if and only if*

$$\mathrm{Hterm}(\mathscr{M}) = \bigcup_{t \in \mathrm{Hterm}(\mathscr{G})} T(\mathbf{x})t,$$

where $\mathrm{Hterm}(\mathscr{G}) := \{Hterm(g) : 0 \neq g \in \mathscr{G}\}$. *Moreover, for every $f \in F[\mathbf{x}]^p$, the polynomials $\alpha_1, \dots, \alpha_n \in F[\mathbf{x}]$ and $q \in F[\mathbf{x}]_T^p$ such that*

$$f = \alpha_1 g_1 + \cdots + \alpha_n g_n + q$$

are algorithmically computable (using the reduction algorithm [3]).

Informally speaking, the previous proposition tells that, if $\mathscr{G}$ is a Gröbner basis of $\mathscr{M}$, the $\mathrm{Hterm}(\mathscr{M})$ can be computed directly from the finite set $\mathrm{Hterm}(\mathscr{G})$ and that the decomposition (1) is computable.

When we deal with difference equations operating on signals with support in $\mathbf{Z}^r$, we have to manipulate Laurent polynomials, i.e. polynomials in which negative powers are allowed. Formally the Laurent polynomial ring is the subring of the field of fractions of $F[\mathbf{x}]$ defined as follows:

$$F[\mathbf{x}, \mathbf{x}^{-1}] := F\left[x_1, \dots, x_r, x_1^{-1}, \dots, x_r^{-1}\right] := \left\{t^{-1}q : t \in T(\mathbf{x}),\ q \in F[\mathbf{x}]\right\},$$

i.e. the quotient ring of $F[\mathbf{x}]$ w.r.t. the multiplicative system $T(\mathbf{x})$ [8, p. 46]. It is clear that the group $T(\mathbf{x}, \mathbf{x}^{-1}) := \{x_1^{h_1} \cdots x_r^{h_r} : (h_1, \dots, h_r) \in \mathbf{Z}^r\}$ of all the monic monomials in $F[\mathbf{x}, \mathbf{x}^{-1}]$ constitutes a basis of the F-vector space $F[\mathbf{x}, \mathbf{x}^{-1}]$.

The following theorem shows that the Laurent polynomial ring $F[\mathbf{x}, \mathbf{x}^{-1}] = F[x_1, \dots, x_r, x_1^{-1}, \dots, x_r^{-1}]$ is isomorphic to the quotient ring $F[\mathbf{x}, \mathbf{y}]/\mathscr{U}$, where $F[\mathbf{x}, \mathbf{y}] := F[x_1, \ldots, x_r, y_1, \ldots, y_r]$ and $\mathscr{U}$ is the ideal $F[\mathbf{x}, \mathbf{y}]$ generated by the polynomials $x_1 y_1 - 1, \ldots, x_r y_r - 1$.

Define the set $T(\mathbf{x}, \mathbf{y})$ as the set of all the monic monomials in $x_1, \ldots, x_r, y_1, \ldots, y_r$ and its subset S as follows:

$$S := \left\{ x_1^{h_1} \cdots x_r^{h_r} y_1^{k_1} \cdots y_r^{k_r} : h_1 k_1 = \cdots = h_r k_r = 0 \right\}.$$

It is easy to see that the set $\{s + \mathscr{U} : s \in S\}$ constitutes a basis of the F-vector space $F[\mathbf{x}, \mathbf{y}]/\mathscr{U}$.

THEOREM. *$F[\mathbf{x}, \mathbf{x}^{-1}]$ is isomorphic to $F[\mathbf{x}, \mathbf{y}]/\mathscr{U}$.*

Proof. Consider the homomorphism

$$\Psi : F[\mathbf{x}, \mathbf{y}] \to F[\mathbf{x}, \mathbf{x}^{-1}],$$

which is the identity map on the elements of $F[\mathbf{x}]$ and such that

$$\Psi(y_i) = x_i^{-1}, \qquad i = 1, \ldots, r.$$

Ψ is surjective, since the set $F \cup \{x_1, \ldots, x_r, x_1^{-1}, \ldots, x_r^{-1}\}$ is included in $\operatorname{im}\Psi$. We want to show that $\ker \Psi = \mathscr{U}$. It is clear that $\ker \Psi \supseteq \mathscr{U}$.

Let $F[\mathbf{x}, \mathbf{y}]_S$ be the F-subspace of $F[\mathbf{x}, \mathbf{x}^{-1}]$ generated by S. We show that

$$\ker \Psi \cap F[\mathbf{x}, \mathbf{y}]_S = \{0\} \tag{2}$$

by induction on r, the number of the indeterminates. If $r = 0$, then Ψ is the identity map on F and so $\ker \Psi = \{0\}$. Supposing that (2) is true for $r - 1$, then (2) is true also for r. To prove this, let $q \in F[\mathbf{x}, \mathbf{y}]_S$ and $\Psi(q) = 0$. Then

$$q = \sum_{i \geqslant 0} q_{i0} x_r^i + \sum_{j > 0} q_{0j} y_r^j,$$

where $q_{i0}, q_{0j} \in F[\mathbf{x}, \mathbf{y}]_S \cap F[x_1, \ldots, x_{r-1}, y_1, \ldots, y_{r-1}]$. Moreover, since

$$\Psi(q) = \sum_{i \geqslant 0} \Psi(q_{i0}) x_r^i + \sum_{j > 0} \Psi(q_{0j}) x_r^{-j} = 0,$$

we have that $\Psi(q_{i0}) = \Psi(q_{0j}) = 0$ for all i and j, and so, by the inductive hypothesis, $q_{i0} = q_{0j} = 0$ and $q = 0$.

If $q \in \ker \Psi$, then there exist $p \in \mathscr{U}$ and $p' \in F[\mathbf{x}, \mathbf{y}]_S$ such that $q = p + p'$, and so $\Psi(q) = \Psi(p') = 0$. Therefore by (8) $p' = 0$ and $q \in \mathscr{U}$. ∎

The previous isomorphism allows the use of Gröbner basis techniques for solving problems formulated in the Laurent polynomial ring. For our aims, we are interested in the generalization of Proposition 1.

Consider the $F[\mathbf{x}, \mathbf{x}^{-1}]$-module $F[\mathbf{x}, \mathbf{x}^{-1}]^p$, and define its subset $T_p(\mathbf{x}, \mathbf{x}^{-1})$ as follows:

$$\begin{aligned} T_p(\mathbf{x}, \mathbf{x}^{-1}) &:= \bigcup_{i=1}^{p} T(\mathbf{x}, \mathbf{x}^{-1}) e_i \\ &= \left\{ \left(0, \dots, 0, \overset{i}{t}, 0, \dots, 0\right)^T : i = 1, \dots, p, t \in T(\mathbf{x}, \mathbf{x}^{-1}) \right\}. \end{aligned}$$

Consider moreover the $F[\mathbf{x}, \mathbf{y}]$-module $F[\mathbf{x}, \mathbf{y}]^p$, and define its subset $T_p(\mathbf{x}, \mathbf{y})$ as follows:

$$\begin{aligned} T_p(\mathbf{x}, \mathbf{y}) &:= \bigcup_{i=1}^{p} T(\mathbf{x}, \mathbf{y}) e_i \\ &= \left\{ \left(0, \dots, 0, \overset{i}{t}, 0, \dots, 0\right)^T : i = 1, \dots, p, t \in T(\mathbf{x}, \mathbf{y}) \right\}. \end{aligned}$$

Let Ψ_p be the map defined as follows:

$$\begin{aligned} \Psi_p : F[\mathbf{x}, \mathbf{y}]^p &\to F[\mathbf{x}, \mathbf{x}^{-1}]^p \\ (f_1, \dots, f_p) &\mapsto (\Psi(f_1), \dots, \Psi(f_p)), \end{aligned}$$

whose kernel is the submodule $\mathscr{U}^p$. Let

$$S_p := \bigcup_{i=1}^{p} S e_i = \left\{ \left(0, \dots, 0, \overset{i}{s}, 0, \dots, 0\right)^T : i = 1, \dots, p, s \in S \right\},$$

and moreover let Φ_p be the restriction of Ψ_p to the subspace $F[\mathbf{x}, \mathbf{y}]_{S_p}^p$ of $F[\mathbf{x}, \mathbf{y}]^p$ generated by S_p. It is easy to see that Φ_p is a F-vector space isomorphism between $F[\mathbf{x}, \mathbf{y}]_{S_p}^p$ and $F[\mathbf{x}, \mathbf{x}^{-1}]^p$. The map Φ_p provides also a one to one relation between S_p and $T_p(\mathbf{x}, \mathbf{x}^{-1})$. Therefore, if we have a total admissible ordering in $T_p(\mathbf{x}, \mathbf{y})$, then a total ordering is induced in $T_p(\mathbf{x}, \mathbf{x}^{-1})$. This allows us to define $\text{Hterm}(f)$ for each $f \in F[\mathbf{x}, \mathbf{x}^{-1}]^p$, as we did previously in the standard case. It can be easily seen the $\text{Hterm}(f) = \Phi_p(\text{Hterm}(\Phi_p^{-1}(f)))$.

Note that, if $\bar{f} \in F[\mathbf{x}, \mathbf{y}]^p$ and $f \in F[\mathbf{x}, \mathbf{x}^{-1}]^p$, then no multiplications are needed for the computation of $\Psi_p(\bar{f})$ and of $\Phi_p^{-1}(f)$, and so they can be considered zero complexity operations.

LEMMA 2. *Let $\mathscr{M}$ be a submodule of $F[\mathbf{x}, \mathbf{x}^{-1}]^p$ and $\bar{\mathscr{M}} = \Psi_p^{-1}(\mathscr{M})$. If $\bar{T}$ is a subset of $T_p(\mathbf{x}, \mathbf{y})$ such that*

$$F[\mathbf{x}, \mathbf{y}]^p = \bar{\mathscr{M}} \oplus F[\mathbf{x}, \mathbf{y}]^p_{\bar{T}},$$

then we have that

$$F[\mathbf{x}, \mathbf{x}^{-1}]^p = \mathscr{M} \oplus F[\mathbf{x}, \mathbf{x}^{-1}]^p_T, \tag{3}$$

where $T = \Psi_p(\bar{T})$.

Proof. Let $f \in F[\mathbf{x}, \mathbf{x}^{-1}]^p$. Then, since $F[\mathbf{x}, \mathbf{y}]^p = \bar{\mathscr{M}} \oplus F[\mathbf{x}, \mathbf{y}]^p_{\bar{T}}$, there exist $\bar{f}_1 \in \bar{\mathscr{M}}$ and $\bar{f}_2 \in F[\mathbf{x}, \mathbf{y}]^p_{\bar{T}}$ such that $\Phi^{-1}(f) = \bar{f}_1 + \bar{f}_2$ and so $f = \Psi(\bar{f}_1) + \Psi(\bar{f}_2)$, where $\Psi(\bar{f}_1) \in \mathscr{M}$ and $\Psi(\bar{f}_2) \in F[\mathbf{x}, \mathbf{x}^{-1}]^p_T$. Suppose that $f \in \mathscr{M} \cap F[\mathbf{x}, \mathbf{x}^{-1}]^p_T$. Since $\Psi_p(F[\mathbf{x}, \mathbf{y}]^p_{\bar{T}}) = F[\mathbf{x}, \mathbf{x}^{-1}]^p_T$, then there exists $\bar{f} \in F[\mathbf{x}, \mathbf{y}]^p_{\bar{T}}$ such that $f = \Psi_p(\bar{f})$, and so, since $f \in \mathscr{M}$, we have $\bar{f} \in \bar{\mathscr{M}}$. Consequently $\bar{f} = 0$ and $f = 0$. ∎

If a submodule $\mathscr{M}$ of $F[\mathbf{x}, \mathbf{x}^{-1}]^p$ is generated by $p_1, \ldots, p_l$, then a set of generators of the submodule $\bar{\mathscr{M}} = \Psi_p^{-1}(\mathscr{M})$ of $F[\mathbf{x}, \mathbf{y}]^p$ is constituted by $(x_i y_i - 1)e_j$, $i = 1, \ldots, r$, $j = 1, \ldots, p$ and by $\Phi^{-1}(p_1), \ldots, \Phi^{-1}(p_l)$. A Gröbner basis $\bar{\mathscr{G}} = \{\bar{g}_1, \ldots, \bar{g}_n\}$ of $\bar{\mathscr{M}}$ can be provided by standard Gröbner basis algorithms. Moreover, such algorithms provide also the polynomials $\bar{\delta}_{i1}, \ldots, \bar{\delta}_{il} \in F[\mathbf{x}, \mathbf{y}]$ and the polynomial vector $\bar{u} \in \mathscr{U}^p$ such that

$$\bar{g}_i = \bar{\delta}_{i1}\Phi^{-1}(p_1) + \cdots + \bar{\delta}_{il}\Phi^{-1}(p_l) + \bar{u},$$

for all $i = 1, \ldots, n$. Therefore Lemma 2, together with Proposition 1, provides a method for constructing from $\bar{\mathscr{G}}$ a set T which satisfies the condition (3).

If we let $g_i = \Psi_p(\bar{g}_i)$ and $\delta_{ij} = \Psi(\bar{\delta}_{ij})$, then $\mathscr{G} = \{g_1, \dots, g_n\}$ constitutes a set of generators of $\mathscr{M}$ and

$$g_i = \delta_{i1} p_1 + \cdots + \delta_{il} p_l$$

for all $i = 1, \dots, n$. The set $\mathscr{G}$ will be called a Gröbner basis of $\mathscr{M}$. This name is motivated by the following properties of $\mathscr{G}$:

PROPOSITION 2. *Let $\mathscr{G} = \{g_1, \dots, g_n\}$ be a Gröbner basis of $\mathscr{M}$, and* $\operatorname{Hterm}(\mathscr{G}) := \{\operatorname{Hterm}(g) : 0 \neq g \in \mathscr{G}\}$. *Define for all $x_1^{h_1} \cdots x_r^{h_r} e_i \in T_p(\mathbf{x}, \mathbf{x}^{-1})$ the set*

$$H\left(x_1^{h_1} \cdots x_r^{h_r} e_i\right) := \left\{x_1^{k_1} \cdots x_r^{k_r} : (k_1, \dots, k_r) \in \mathbf{Z}^r, \right.$$
$$\left. h_1 k_1 \geqslant 0, \dots, h_r k_r \geqslant 0\right\}.$$

Then the set

$$T = T_p(\mathbf{x}, \mathbf{x}^{-1}) \setminus \bigcup_{d \in \operatorname{Hterm}(\mathscr{G})} H(d) d$$

satisfies the decomposition (3). *Moreover, for every $f \in F[\mathbf{x}, \mathbf{x}^{-1}]^p$, the polynomials $\alpha_1, \dots, \alpha_n \in F[\mathbf{x}, \mathbf{x}^{-1}]$ and the polynomial vector $q \in F[\mathbf{x}, \mathbf{x}^{-1}]_T^p$ such that*

$$f = \alpha_1 g_1 + \cdots + \alpha_n g_n + q$$

are algorithmically computable.

Proof. First we want to show that $(x_i y_i - 1)e_j$, $i = 1, \dots, r$, $j = 1, \dots, p$, and $\Phi^{-1}(g_1), \dots, \Phi^{-1}(g_n)$ constitutes a Gröbner basis of $\bar{\mathscr{M}}$, which we denote by $\hat{\mathscr{G}}$. Suppose that $\bar{\mathscr{G}} = \{\bar{g}_1, \dots, \bar{g}_n\}$ is the Gröbner basis of $\bar{\mathscr{M}} = \Psi^{-1}(\mathscr{M})$ such that $g_\nu = \Psi(\bar{g}_\nu)$ for all $\nu = 1, \dots, n$. If $\operatorname{Hterm}(\bar{g}_\nu) \notin S_p$, then there exist $(x_i y_i - 1)e_j \in \hat{\mathscr{G}}$ and $v \in T(\mathbf{x}, \mathbf{y})$ such that $\operatorname{Hterm}(\bar{g}_\nu) = v \operatorname{Hterm}((x_i y_i - 1)e_j) = v x_i y_i e_j$. If $\operatorname{Hterm}(\bar{g}_\nu) \in S_p$, then it is easy to see that $\operatorname{Hterm}(\bar{g}_\nu) = \operatorname{Hterm}(\Phi^{-1}(g_\nu))$. Therefore we have that

$$\bigcup_{t \in \operatorname{Hterm}(\hat{\mathscr{G}})} T(\mathbf{x}, \mathbf{y}) t = \bigcup_{t \in \operatorname{Hterm}(\bar{\mathscr{G}})} T(\mathbf{x}, \mathbf{y}) t = \operatorname{Hterm}(\bar{\mathscr{M}}),$$

and so $\hat{\mathscr{G}}$ is a Gröbner basis by Proposition 1.

Consequently $\bar{T} := T_p(\mathbf{x}, \mathbf{y}) \setminus \mathrm{Hterm}(\mathscr{M}) \subseteq S_p$ and so $\Psi_p(\bar{T}) = \Phi_p(\bar{T})$. We want to show that $\Phi_p(\bar{T}) = T$. If $t \notin \Phi_p(\bar{T})$, then $\Phi_p^{-1}(t) \in \mathrm{Hterm}(\mathscr{M}) \cap S_p$ and so there exist $g_i \in \mathscr{G}$ and $v \in T(\mathbf{x}, \mathbf{y})$ such that $\Phi_p^{-1}(t) = v\ \mathrm{Hterm}(\Phi_p^{-1}(g_i))$. Since $\Phi_p^{-1}(t) \in S_p$, then $v \in S$ and $\Phi(v) \in H(\Phi_p(\mathrm{Hterm}(\Phi_p^{-1}(g_i))) = H(\mathrm{Hterm}(g_i))$. Therefore we have that $t = \Phi_p(v)\Phi_p(\mathrm{Hterm}(\Phi_p^{-1}(g_i))) = \Phi_p(v)\,\mathrm{Hterm}(g_i)$ and so

$$t \in \bigcup_{d \in \mathrm{Hterm}(\mathscr{G})} H(d)d.$$

On the other side, suppose that there exist $g_i \in \mathscr{G}$ and $v \in H(\mathrm{Hterm}(g_i))$ such that $t = v\ \mathrm{Hterm}(g_i)$. Then $\Phi^{-1}(t) = \Phi^{-1}(v\ \mathrm{Hterm}(g_i)) = \Phi(v)\,\mathrm{Hterm}(\Phi_p^{-1}(g_i))$ and so $t \notin \Phi_p(\bar{T})$.

Finally, if f is a Laurent polynomial vector in $F[\mathbf{x}, \mathbf{x}^{-1}]^p$ and if $\bar{f} = \Phi^{-1}(f)$, then by Proposition 1 we can compute the decomposition

$$\bar{f} = \bar{\alpha}_1 \Phi_p^{-1}(g_1) + \cdots + \bar{\alpha}_n \Phi_p^{-1}(g_n) + \bar{u} + \bar{q},$$

where $\bar{u} \in \mathscr{U}^p$ and $\bar{q} \in F[\mathbf{x}, \mathbf{x}^{-1}]_{\bar{T}}^p$. Consequently we have

$$f = \alpha_1 g_1 + \cdots + \alpha_n g_n + q,$$

where $\alpha_i = \Psi(\bar{\alpha}_i)$ for all $i = 1, \ldots, n$, and so $q = \Psi_p(\bar{q}) \in F[\mathbf{x}, \mathbf{x}^{-1}]_T^p$ and the decomposition (3) is computable. ∎

The decomposition (3) is the key result we need for solving the Cauchy problem. The previous proposition tells moreover that such a decomposition is computable, and it suggests how to construct the set T directly from the Gröbner basis $\mathscr{G}$.

3. MULTIDIMENSIONAL LINEAR SHIFT-INVARIANT SYSTEMS

The behavioral approach provides a correct framework for stating and solving the Cauchy problem in the context of multidimensional discrete systems. A complete treatment of this approach can be found in [1–3]. Here we confine ourselves to a brief introduction.

Let F be a field, and $A = F^{\mathbf{Z}^r}$ be the F-vector space of all functions from $\mathbf{Z}^r$ to F, called the *signal space*. For $i = 1, \ldots, r$, define the operators

$$\sigma_i : A \to A : w \mapsto \sigma_i w$$

as follows:

$$(\sigma_i w)(t_1, \dots, t_i, \dots, t_r) = w(t_1, \dots, t_i + 1, \dots, t_r).$$

These operators are linear and invertible, and they commute. Consider the F-algebra of all the linear maps from A to A, and denote by $F[\sigma_1, \dots, \sigma_r, \sigma_1^{-1}, \dots, \sigma_r^{-1}] =: F[\sigma, \sigma^{-1}]$ the subalgebra generated by $\sigma_1, \dots, \sigma_r, \sigma_1, \dots, \sigma_r^{-1}$. Note that this commutative algebra is isomorphic to the Laurent polynomial algebra $F[x_1, \dots, x_r, x_1^{-1}, \dots, x_r^{-1}] = F[\mathbf{x}, \mathbf{x}^{-1}]$.

Let $R \in F[\sigma, \sigma^{-1}]^{l \times q}$ be any matrix with entries r_{ij}, $i = 1, \dots, l$, $j = 1, \dots, q$, in $F[\sigma, \sigma^{-1}]$. In the sequel we will associate to R the linear operator from A^q into A^l that maps $w = (w_1, \dots, w_q)^T$ in $v = (v_1, \dots, v_l)^T = Rw$ as follows:

$$v_i = \sum_{j=1}^{q} r_{ij} w_j, \qquad i = 1, \dots, l.$$

As we previously said, in the behavioral approach a dynamical system Σ is defined as a triple

$$\Sigma = (T, W, \mathscr{B}),$$

where T is the time set, W is the signal alphabet, and $\mathscr{B} \subseteq W^T$ is the behavior. In this paper we will consider systems with $T = \mathbf{Z}^r$, $W = F^q$, and $\mathscr{B} = \ker R$ for some $R \in F[\sigma, \sigma^{-1}]^{l \times q}$ and $l \in \mathbf{N}$. In this case the behavior $\mathscr{B}$ coincides with the set of solutions of a difference equation represented by the polynomial matrix R.

We will give some results about linear shift-invariant dynamical systems that can be found in [1–3]. The proofs we will present exploit the duality theory of vector spaces [9].

Consider the nondegenerate bilinear map

$$\langle \cdot , \cdot \rangle : F[\mathbf{x}, \mathbf{x}^{-1}]^q \times A^q \to F$$

defined as follows. If $w = (w_1, \dots, w_q)^T$ is any signal in A^q, then for all $t = x_1^{h_1} \cdots x_r^{h_r} e_i \in T_q(\mathbf{x}, \mathbf{x}^{-1})$ we let

$$\langle t, w \rangle := w_i(h_1, \dots, h_r).$$

Since $T_q(\mathbf{x}, \mathbf{x}^{-1})$ is a F-basis of $F[\mathbf{x}, \mathbf{x}^{-1}]^q$, this definition uniquely extends to all $F[\mathbf{x}, \mathbf{x}^{-1}]^q$. In this way A^q can be considered as a dual space of $F[\mathbf{x}, \mathbf{x}^{-1}]^p$. More precisely, A^q is isomorphic to the space of all the F-linear functionals defined on $F[\mathbf{x}, \mathbf{x}^{-1}]^q$. Let R be a matrix in $F[\sigma, \sigma^{-1}]^{l \times q}$, and $\overline{R}$ its isomorphic image in $F[\mathbf{x}, \mathbf{x}^{-1}]^{l \times q}$. It is easy to see that

$$\begin{array}{llll} R: & A^q & \to A^l & : w \mapsto Rw \\ \overline{R}^T: & F[\mathbf{x}, \mathbf{x}^{-1}]^l & \to F[\mathbf{x}, \mathbf{x}^{-1}]^q & : f \mapsto \overline{R}^T f \end{array}$$

are dual maps, since $\langle f, Rw \rangle = \langle \overline{R}^T f, w \rangle$ for all $f \in F[\mathbf{x}, \mathbf{x}^{-1}]^q$ and $w \in A^q$.

First it can be seen that the behavior $\mathscr{B} = \ker R$ is uniquely determined by $\operatorname{im} \overline{R}^T$, that is, the submodule of $F[\mathbf{x}, \mathbf{x}^{-1}]^q$ generated by the columns of $\overline{R}^T$. This is the consequence of the relations

$$\ker R = (\operatorname{im} \overline{R}^T)^{\perp} := \{w \in A^q : \langle f, w \rangle = 0 \; \forall f \in \operatorname{im} \overline{R}^T\},$$

$$\operatorname{im} \overline{R}^T = (\ker R)^{\perp} := \{f \in F[\mathbf{x}, \mathbf{x}^{-1}]^q : \langle f, w \rangle = 0 \; \forall w \in \ker R\}.$$

Consider the system $\Sigma = (\mathbf{Z}^r, F^q, \ker R)$ with $R \in F[\sigma, \sigma^{-1}]^{l \times q}$, $p :=$ rank R, and $m := q - p$, and consider moreover any permutation of $w_1, \ldots, w_q$ and of the corresponding columns of R such that the first p columns of R become independent. Partition R as $[-P \mid Q]$ and w as $(y, u)^T$ with $P \in F[\sigma, \sigma^{-1}]^{l \times p}$, $Q \in F[\sigma, \sigma^{-1}]^{l \times m}$, $y \in A^p$, and $u \in A^m$. Therefore we have that

$$\ker R = \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in A^q : Py = Qu \right\}.$$

The equation $Py = Qu$ is called an input-output representation of Σ. In fact is can be shown that for every $u \in A^m$ there exists $y \in A^p$ such that $(y, u)^T \in \ker R$. Actually, if $\overline{P}$ is the isomorphic image of P in $F[\mathbf{x}, \mathbf{x}^{-1}]^{l \times p}$ and $\overline{Q}$ is the isomorphic image of Q in $F[\mathbf{x}, \mathbf{x}^{-1}]^{l \times m}$, then by the rank conditions we have that $\ker \overline{P}^T \subseteq \ker \overline{Q}^T$ and so $\operatorname{im} P \supseteq \operatorname{im} Q$. Since the signal u can be fixed arbitrarily, it is called a free variable or an input, while y is called an output.

When the input u is fixed in a input-output representation $Py = Qu$, the output y has still some degrees of freedom. Actually there exist subsets $G_i \subseteq \mathbf{Z}^r$, $i = 1, \ldots, p$, such that there exists a unique $y = (y_1, \ldots, y_p)^T \in$

A^p satisfying

$$y_i|_{G_i} = z_i, \qquad i = 1, \dots, p, \tag{4}$$
$$Py = Qu$$

for any given $u \in A^m$ and $z_i \in F^{G_i}$. The restrictions $y_i|_{G_i} = z_i$ are called initial conditions of y. The problem of finding the set G_i and of constructing an algorithm which gives y from the initial conditions z_i and from the input u is called the Cauchy problem (see [3]).

In [1, 2] a solution of the Cauchy problem is proposed when $r = 2$. In the following section we will present a new method for solving the Cauchy problem in the general case. This solution is based on the properties of Laurent polynomial submodules shown in Section 2 and follows the same philosophy proposed by Oberst for solving the analogous problem for systems over $\mathbf{N}^r$.

4. SOLUTION OF THE CAUCHY PROBLEM

Given the system $\Sigma = (\mathbf{Z}^r, F^q, \ker R)$, $R \in F[\sigma, \sigma^{-1}]^{l \times q}$, and one of its input-output representations

$$Py = Qu, \tag{5}$$

where $P \in F[\sigma, \sigma^{-1}]^{l \times p}$, $Q \in F[\sigma, \sigma^{-1}]^{l \times m}$, the Cauchy problem must be solved in two steps:

(a) find subsets $G_i \subseteq \mathbf{Z}^r$, $i = 1, \dots, p$, such that there exists a unique $y = (y_1, \dots, y_p)^T \in A^p$ satisfying (4) for any given $u \in A^m$ and $z_i \in F^{G_i}$;

(b) given $u \in A^m$ and $z_i \in F^{G_i}$, $i = 1, \dots, p$, compute the unique output $y \in A^p$ satisfying (4).

Let $\bar{P}$ be the isomorphic image of P in $F[\mathbf{x}, \mathbf{x}^{-1}]^{l \times p}$, and $\bar{Q}$ be the isomorphic image of Q in $F[\mathbf{x}, \mathbf{x}^{-1}]^{l \times m}$. Let $\mathscr{M} := \operatorname{im} \bar{P}^T$ be the submodule generated by the columns of $\bar{P}^T$. By the method we presented in Section 2 we compute a Gröbner basis $\mathscr{G} = \{g_1, \dots, g_n\}$ of $\mathscr{M}$, the Laurent polynomial matrix $\bar{X}$ such that

$$\bar{P}_g^T := [g_1, \dots, g_n] = \bar{P}^T \bar{X},$$

and a subset T of $T_p(\mathbf{x}, \mathbf{x}^{-1})$ satisfying (3). It is easy to see that, if we let $\overline{Q}_g := \overline{X}^T\overline{Q}$, then we have that

$$\operatorname{im}\begin{bmatrix} -\overline{P}^T \\ \overline{Q}^T \end{bmatrix} = \operatorname{im}\begin{bmatrix} -\overline{P}_q^T \\ \overline{Q}_g^T \end{bmatrix},$$

and so, if P_g is the isomorphic image of $\overline{P}_g$ in $F[\sigma, \sigma^{-1}]^{n\times p}$ and Q_g is the isomorphic image of $\overline{Q}_g$ in $F[\sigma, \sigma^{-1}]^{n\times m}$, then the input-output representation

$$P_g y = Q_g u$$

represents the same behavior as the input-output representation (5).

(a): First we will show that sets G_i in (4) can be obtained from T as follows:

$$G_i := \left\{(h_1, \dots, h_r) \in \mathbf{Z}^r : x_1^{h_1} \cdots x_r^{h_r} e_i \in T\right\}. \tag{6}$$

Actually, since we have that

$$F[\mathbf{x}, \mathbf{x}^{-1}]^p = \mathcal{M} \oplus F[\mathbf{x}, \mathbf{x}^{-1}]_T^p,$$

we obtain that (see [9, p. 73])

$$A^p = \mathcal{M}^\perp \oplus \left(F[\mathbf{x}, \mathbf{x}^{-1}]_T^p\right)^\perp.$$

It is easy to see that $(F[\mathbf{x}, \mathbf{x}^{-1}]_T^p)^\perp$ is the subset $A^p(G_1, \dots, G_p)$ of A^p defined as follows:

$$A^p(G_1, \dots, G_p) := \left\{y = (y_1, \dots, y_p) \in A^p : y_i|_{G_i} = 0,\ i = 1, \dots, p\right\},$$

where $G_1, \dots, G_p$ are the subsets of $\mathbf{Z}^r$ defined in (6). Therefore A^p splits as

$$A^p = \ker P \oplus A^p(G_1, \dots, G_p). \tag{7}$$

Now we show that (7) implies the existence and uniqueness of the solution y of (4) for a given $u \in A^m$ and $z_i \in F^{G_i}$, $i = 1, \dots, p$. Since u is a vector of free variables, there exists $v \in A^p$ such that $Pv = Qu$. Let w be any signal in A^p such that $w_i|_{G_i} = z_i$. According to the decomposition (7) we have that $v = v' + v''$ and $w = w' + w''$, where $v', w' \in \ker P$ and $v'', w'' \in A^p(G_1, \dots, G_p)$. Consequently the signal $y = w' + v''$ is the solution of (4), since

$$y_i|_{G_i} = w_i'|_{G_i} + v_i''|_{G_i} = w_i|_{G_i} = x_i, \qquad i = 1, \dots, p,$$

$$Py = Pw' + Pv'' = Pz = Qu.$$

Finally suppose that both y and $\hat{y}$ satisfy (4). Then $\delta := y - \hat{y}$ satisfies the following conditions:

$$\delta_i|_{G_i} = 0, \qquad i = 1, \dots, p,$$

$$P\delta = 0.$$

Therefore δ is included in $\ker P \cap A^p(G_1, \dots, G_p)$, and hence $\delta = 0$.

(b): Given $h = (h_1, \dots, h_r) \in \mathbf{Z}^r$ and $i \in \{1, \dots, p\}$, the polynomials $\beta_1, \dots, \beta_n \in F[\mathbf{x}, \mathbf{x}^{-1}]$ and the polynomial vector $q \in F[\mathbf{x}, \mathbf{x}^{-1}]_T^p$ such that

$$x_1^{h_1} \cdots x_r^{h_r} e_i = \beta_1 g_1 + \cdots + \beta_n g_n + q$$

are effectively computable. If $\beta^T = (\beta_1, \dots, \beta_n)$, then

$$y_i(h) = \langle x_1^{h_1} \cdots x_r^{h_r} e_i, y \rangle = \langle \bar{P}_g^T \beta, y \rangle + \langle q, y \rangle = \langle \beta, P_g y \rangle + \langle q, y \rangle$$

$$= \langle \beta, Q_g u \rangle + \langle q, y \rangle = \langle \bar{Q}_g^T \beta, u \rangle + \langle q, y \rangle.$$

Since u and $y_i|_{G_i}$, $i = 1, \dots, p$, are given, each term of the sum can be computed directly, applying the definition of $\langle \cdot, \cdot \rangle$. Clearly, the signal y is the sum of two components, one depending only on the inputs (forced response) and the other depending on the initial condition $y_i|_{G_i}$ (free response).

REMARK. Note that our method can be easily applied to multidimensional systems defined on any time set which is isomorphic to $\mathbf{Z}^r \times \mathbf{N}^s$, for some positive integers r and s. There is however a limitation on this way of solving multidimensional linear difference equations. In fact, whereas the choice of the free variable structure is completely free, there is no freedom in the choice of the initial condition structure. In particular, our method does not solve multidimensional difference equations when boundary conditions are given.

5. NUMERICAL EXAMPLE

Let $A = R^{\mathbf{Z}^2}$, and consider the system $\Sigma = (\mathbf{R}^3, \mathbf{Z}^2, \mathscr{B})$, where $\mathscr{B} = \ker R$ and where

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \end{bmatrix}$$

with

$$r_{11} = x_2^{-1} + 2x_1x_2^{-1} + x_2,$$

$$r_{21} = x_1x_2^2 - 1,$$

$$r_{12} = x_1^{-1}x_2 - x_1,$$

$$r_{22} = x_1^2x_2^{-1} + 2 - x_2^{-1},$$

$$r_{13} = x_1^{-1}x_2^{-1} + 2x_2^{-1} - x_1,$$

$$r_{23} = x_2^2 - x_1^{-1} - x_1^2x_2^{-1} - 2 + x_2.$$

We have identified the elements in $F[\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}]$ with the elements in $F[x_1, x_2, x_1^{-1}, x_2^{-1}]$. If

$$P = \begin{bmatrix} r_{11} & r_{21} \\ r_{21} & r_{22} \end{bmatrix}, \qquad Q = \begin{bmatrix} r_{13} \\ r_{23} \end{bmatrix},$$

then we have the rank R = rank $P = 2$. More precisely, we have that

$$Q = P\begin{bmatrix} x_1^{-1} \\ -1 \end{bmatrix}.$$

Therefore we have that

$$Py = Qu, \tag{8}$$

where $y \in A^2$ and $y \in A$, is an input-output representation of $\mathscr{B}$.

We choose the following term ordering in $T_2(x_1, x_2, y_1, y_2)$:

$$x_1^{h_1} x_2^{h_2} y_1^{h_3} y_2^{h_4} e_i <_T x_q^{k_1} x_2^{k_2} y_1^{k_3} y_2^{k_4} e_j$$

if and only if one of the following conditions holds:

1. $\Sigma_{\nu=1}^{4} h_\nu < \Sigma_{\nu=1}^{4} k_\nu$.
2. $\Sigma_{\nu=1}^{4} h_\nu = \Sigma_{\nu=1}^{4} k_\nu$ and $\exists l : h_\nu + k_\nu$ if $\nu < l$ and $h_l < k_l$.
3. $h_\nu = k_\nu$ for all $\nu = 1, 2, 3, 4$ and $i < j$.

We compute now a Gröbner basis of the submodule $\mathscr{M}$ of $F[x_1, x_2, x_1^{-1}, x_2^{-1}]^2$ generated by $(r_{11}, r_{12})^T$ and $(r_{21}, r_{22})^T$. We obtain the following equivalent input-output representation:

$$P_g y = Q_g u,$$

where

$$P_g = \begin{bmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \\ p_{3,1} & p_{3,2} \\ p_{4,1} & p_{4,2} \\ p_{5,1} & p_{5,2} \\ p_{6,1} & p_{6,2} \\ p_{7,1} & p_{7,2} \\ p_{8,1} & p_{8,2} \\ p_{9,1} & p_{9,2} \\ p_{10,1} & p_{10,2} \end{bmatrix}, \qquad Q_g = P_g\begin{bmatrix} x_1^{-1} \\ -1 \end{bmatrix}$$

with

$$
\begin{aligned}
p_{1,1} &= x_2 x_1^{-1} - x_1, \\
p_{1,2} &= x_1^2 x_2^{-1} - x_2^{-1} + 2, \\
p_{2,1} &= x_2 x_1^{-2} - 1, \\
p_{2,2} &= x_1 x_2^{-1} - x_1^{-1} x_2^{-1} + 2 x_1^{-1}, \\
p_{3,1} &= x_1 x_2^{-2} + \tfrac{1}{2} x_2^{-2} + \tfrac{1}{2}, \\
p_{3,2} &= \tfrac{1}{2} x_1 x_2 - \tfrac{1}{2} x_2^{-1}, \\
p_{4,1} &= x_2^2 x_1^{-1} - x_1 x_2, \\
p_{4,2} &= x_1^2 + 2 x_2 - 1, \\
p_{5,1} &= x_2 x_1^{-1} + x_1^{-1} x_2^{-1} + 2 x_2^{-1}, \\
p_{5,2} &= x_2^2 - x_1^{-1}, \\
p_{6,1} &= x_1^{-1} x_2^{-2} + 2 x_2^{-2} + x_1^{-1}, \\
p_{6,2} &= -x_1^{-1} x_2^{-1} + x_2, \\
p_{7,1} &= x_1^{-2} - x_2^{-1}, \\
p_{7,2} &= x_1 x_2^{-2} - x_1^{-1} x_2^{-2} + 2 x_1^{-1} x_2^{-1}, \\
p_{8,1} &= - x_1^2 x_2 + x_2^2, \\
p_{8,2} &= x_1^3 + 2 x_1 x_2 - x_1, \\
p_{9,1} &= - x_1^{-3} + x_1^{-1} x_2^{-1}, \\
p_{9,2} &= x_1^{-2} x_2^{-2} - 2 x_1^{-2} x_2^{-1} - x_2^{-2}, \\
p_{10,1} &= x_1^2 x_2^3 + 2 x_1^3 x_2^{-1} - x_2^4 + x_1^2 x_2 + x_1^2 x_2^{-1} \\
&\quad - 2 x_1 x_2^{-1} + 2 x_2^2 + 4 x_1 - x_2 - x_2^{-1} + 2, \\
p_{10,2} &= - x_1^2 - 2 x_2 + 1.
\end{aligned}
$$

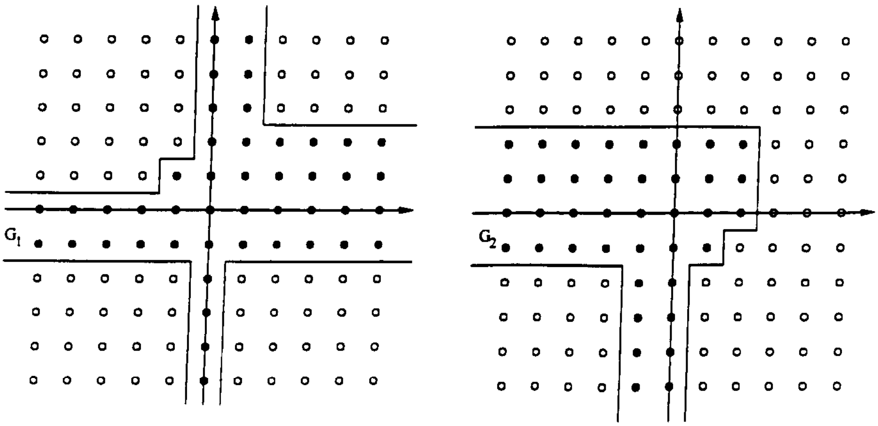

FIG. 1. The sets G_1 and G_2.

Figure 1 shows the subsets G_1 and G_2 where the outputs y_1 and y_2 can be given arbitrarily.

Suppose that $u(i,j) = 0$ for all $i,j \in \mathbf{Z}^2$. Suppose moreover that $y_1(i,j) = 1$ for all $(i,j) \in G_1$ and that $y_2(i,j) = i - j$ for all $(i,j) \in G_2$. We want to compute $y_1(-2,1)$. By Gröbner basis algorithm we obtain that

$$\begin{bmatrix} x_1^{-2}x_2 \\ 0 \end{bmatrix} = P_g^T \beta + \begin{bmatrix} -x_2 \\ x_1 x_2^{-1} - x_1^{-1}x_2^{-1} + 2x_1^{-1} \end{bmatrix},$$

where $\beta = (0, x_2, 0, 0, 0, 0, 0, 0, 0, 0)^T$. Then

$$y_1(-2,2) = \langle Q_g \beta, u \rangle + \left\langle \begin{bmatrix} -x_2 \\ x_1 x_2^{-1} - x_1^{-1}x_2^{-1} + 2x_1^{-1} \end{bmatrix} \right\rangle = -1$$

REFERENCES

1 P. Rocha, Structure and Representation of 2-D Systems, Ph.D. Thesis, Univ. of Groningen, The Netherlands, 1990.

2 P. Rocha and J. C. Willems, Canonical computational forms for ar 2-d systems, *Multidimens. Systems Signal Process.* 2:251–278 (1990).

3 U. Oberst, Multidimensional constant linear systems, *Acta Appl. Math.* 20:1–175 (1990).

4 B. Buchberger, Gröbner basis: An algorithmic method in polynomial ideal theory, in *Multidimensional Systems Theory* (N. K. Bose, Ed.), D. Reidel, 1985, pp. 184–232.
5 E. Fornasini, A note on output feedback stabilizability of multivariable 2-d systems, *Systems Control Lett.* 10:45–50 (1988).
6 J. C. Willems, From time series to linear system, part i: Finite dimensional linear time invariant systems, *Automatica* 22:561–580 (1986).
7 H. M. Möller and F. Mora, New constructive methods in classical ideal theory, *J. Algebra* 100:138–178 (1986).
8 O. Zarisky and P. Samuel, *Commutative Algebra*, Van Nostrand, 1958.
9 W. Greub, *Linear Algebra*, Springer-Verlag, 1975.

Received 6 July 1992; final manuscript accepted 29 January 1993